The $P - Core$ and some related computational procedures

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Abstract. In many of the extensions of the core of a game with transferable utility to non-balanced games the subjacent idea is to construct first a related balanced game, and then, to take its core as an extended core notion for the original non-balanced game. The ε-core and the aspiration core are two examples of this methodology. While the balanced game related to the ε-core is gotten by reducing in a positive amount ε the value of any non-empty coalition different from the grand coalition, the balanced game related to the aspiration core is obtained by increasing the value of the grand coalition by an appropriate quantity. In both cases, the balanced game obtained is not an orthogonal projection of the original game onto the cone of balanced games. We propose a new extension which uses, as an auxiliary balanced game, a game which is, indeed, an orthogonal projection of the original game on the cone of balanced games. In this projected game, the value of the grand coalition increases while the value of some of the others coalitions decreases. In this note e explore some computational procedures to get the projected game.

1 Introduction

The core is the most widely accepted solution concept for games with transferable utility (TU-game). The well-known Shapley-Bondareva Theorem (Bondareva [2], Shapley [7]) characterizes the class of games with non-empty core as the class of balanced games. Shapley [7]) also shows that the set of balanced games is a full dimensional cone. The fact that the core may be the empty set has motivated the study of several core-type solutions for other classes of games containing that of balanced games. Relevant to our purpose are the strong ε-core (Shapley and Shubik [8]) and the aspiration core (Bennett [1]), which are solution concepts defined for any TU-game. These examples illustrate two basic ways that one has to restore the balancedness of a game. Namely, either to decrease the value of intermediate coalitions, or to increase the value of the grand coalition. However, both procedures share a common feature. Each one of them associates first, to any given non-balanced game, another balanced game. Then, they define, as the solution concept for the non-balanced game, the core of the associated balanced game. In the case of the aspiration core, and for a very specific ε-value in the case of the strong ε-core as well, the associated balanced game is a non-orthogonal projection of the non-balanced game onto the cone of balanced games. In Cesco [5] we propose a new extension of the core to non-balanced games, the P-core, by considering orthogonal projections. The resulting procedure associates, to any non-balanced game, a balanced game where the two aforementioned effects show up simultaneously, that is, the value of the grand coalition increases while the value of some intermediate coalitions decrease (with respect to the original game). Here, we explore some computational procedures to get the projected game for some particular cases. We expect that the ideas developed in these procedures will lead to the obtention of a method to get the projected game in the general case.

2 The cone of balanced games

A *TU*-game is an ordered pair (N, v) where $N = \{1, 2, ..., n\}$ is a finite non-empty set, the set of *players,* v is the characteristic function, which is a real valued function defined on the family of subsets of $N, \mathcal{P}(N)$ satisfying $v(\phi) = 0$. The elements in $\mathcal{P}(N)$ are the *coalitions.*

The set of *pre-imputations* is $E = \{x = (x_1, \ldots, x_n) \in \mathbb{R}^n : \sum_{i \in N} x_i = v(N)\}$, and the set of *imputations* is $A = \{x \in E : x_i \geq v(\{i\})$ for all $i \in N\}$.

Given a pre-imputation x and a coalition S in a game (N, v), the *excess* of the coalition S with respect to x is $e(S, x) = v(S) - x(S)$, where $x(S) = \sum_{i \in S} x_i$ if $S \neq \phi$ and 0 otherwise. The *core* of (N, v) is the set $C(N, v) = \{x \in E : e(S, x) \leq 0$ for all $S \in \mathcal{P}(N)\}$.

The core of a game may be the empty set. The Shapley-Bondareva theorem (Bondareva [2], Shapley [7]) characterizes the sub-class of *TU*-games with non-empty core. There, the notion of balanced subfamily of coalitions plays a key role. A non-empty family of coalitions $\mathcal{B}$ is a *balanced family* if there exists a set of positive numbers $\lambda^{\mathcal{B}} = (\lambda_S^{\mathcal{B}})_{S \in \mathcal{B}}$, the *balancing weights*, such that $\sum\limits_{S \in \mathcal{B}(i)} \lambda_S^{\mathcal{B}} = 1$ for all $i \in N$, where $\mathcal{B}(i) = \{S \in B : i \in S\}$. The quantity $w(\mathcal{B}, \lambda_S^{\mathcal{B}}, v) = \sum\limits_{S \in \mathcal{B}} \lambda_S^{\mathcal{B}} v(S)$ is the *worth* of the balanced family $\mathcal{B}$ with respect to a set of balancing weights $\lambda^{\mathcal{B}}$ in a game (N, v). A minimal balanced family is one including no other proper balanced subfamily, and it has a unique set of balancing weights (Shapley [7]). In this case, sometimes we will use the simpler notation $w(\mathcal{B}, v)$ for $w(\mathcal{B}, \lambda_S^{\mathcal{B}}, v)$. Let $\mathcal{B}(N) = \{\mathcal{B} : \mathcal{B}$ is a balanced family such that $N \notin \mathcal{B}\}$, and $\mathcal{M}(N) = \{\mathcal{B} \in \mathcal{B}(N) : \mathcal{B}$ is a minimal balanced family$\}$. Then $\mathcal{M}(N) = \{\mathcal{B} : \mathcal{B}$ is a minimal balanced family such that $\mathcal{B} \neq \{N\}\}$.

A game (N, v) is *balanced* if $w(\mathcal{B}, \lambda_S^{\mathcal{B}}, v) \leq v(N)$ for all (minimal) balanced family $\mathcal{B}$ with balancing weights $\lambda^{\mathcal{B}}$. Shapley- Bondareva's theorem states that the core of a $TU-$game is non-empty if and only if the game is balanced.

In the rest of the paper we are going to use the following notation. Given a game (N, v) and $\mathcal{B} \in \mathcal{B}(N)$ $(\mathcal{M}(N))$, $\Delta(\mathcal{B}, \lambda^{\mathcal{B}}, v)$ $(\Delta(\mathcal{B}, v))$ will stand for $w(\mathcal{B}, \lambda^{\mathcal{B}}, v) - v(N)$, and $\Delta(N, v)$ for $w(N, v) - v(N)$. We also define $w(N, v) =$

$\max\{w(\mathcal{B},\lambda^{\mathcal{B}},v) : \mathcal{B} \in \mathcal{B}(N) \cup \{N\}\}$, and $\mathcal{B}^w(N,v) = \{\mathcal{B} \in \mathcal{M}(N) : \Delta(\mathcal{B},v) > 0\}$. $\mathcal{B}^w(N,v)$ is the family of all objecting minimal balanced families in (N,v).

Remark 1. Clearly $\Delta(N,v) \geq 0$, and the game (N,v) is non-balanced if and only if $\Delta(N,v) > 0$. Also, it is non-balanced if and only if $\mathcal{B}^w(N,v) \neq \phi$.

On the other hand, it is well-known that $w(N,v)$ coincides with $\max\{w(\mathcal{B},v) : \mathcal{B} \in \mathcal{M}(N) \cup \{N\}\}$.

Given a family of coalitions $\mathcal{B} \in \mathcal{B}(N)$, and a set $\lambda^{\mathcal{B}}$ of balancing weights for $\mathcal{B}$, let $\delta^{\mathcal{B},\lambda^{\mathcal{B}}}$ be the (2^n-1)-vector with $\delta^{\mathcal{B},\lambda^{\mathcal{B}}}(N) = -1$, $\delta^{\mathcal{B},\lambda^{\mathcal{B}}}(S) = \lambda^{\mathcal{B}}_S$ if $S \in \mathcal{B}$, and $\delta^{\mathcal{B},\lambda^{\mathcal{B}}}(S) = 0$ for any other non-empty coalition S. Whenever $\mathcal{B} \in \mathcal{M}(N)$, we are going to use the simpler notation $\delta^{\mathcal{B}}$ instead of $\delta^{\mathcal{B},\lambda^{\mathcal{B}}}$, since there is no confusion regarding the set of balancing weight for $\mathcal{B}$. If $\mathbb{B}$ denotes the cone generated by the family $\{\delta^{\mathcal{B}} : \mathcal{B} \in \mathcal{M}(N)$ then, the set of balanced TU-games $\mathbb{V}^n = \mathbb{B}^{*}$[3] where $\mathbb{B}^* = \{v \in \mathbb{R}^{2^n-1} : \langle v,\delta\rangle \leq 0 \text{ for all } \delta \in \mathbb{B}\}$ is the polar cone of $\mathbb{B}$. Here, $\langle .,.\rangle$ stands for the usual scalar product in $\mathbb{R}^{(2^n-1)}$. It is well-known that each generating vector $\delta^{\mathcal{B}}$ determines a (2^n-2)-dimensional face of $\mathbb{V}^n$[4] (Shapley [7]).

To end this section, we introduce another extension of the core to non-balanced games. We define the projected core of (N,v) as the core of the orthogonal projection of the game onto $\mathbb{V}^n$. To this end, let (N,v^P) denote the closest balanced game to a given game (N,v). Namely, $(N,v^P) \in \mathbb{V}^n$ is such that $\left\|v^P - v\right\|_2$ is minimum, where $\|.\|$, will stand for the Euclidean norm of $\mathbb{R}^{(2^n-1)}$.

Definition 1. *The P-core $PC(N,v)$ of a given game (N,v) is the core of the associated game (N,v^P).*

It is clear that $PC(N,v) = C(N,v)$ whenever (N,v) is balanced, so the P-core is an extension of the classical core.

Thus, to obtain a point in the core of this projected game of a given non-balanced game (N,v), two steps have to be solved. First, an optimization problem to get the closest game (N,v^P) to (V,n) on $\mathbb{V}^n$. Second, the computation of an imputation in $C(N,v^P)$. The linearly constrained optimization problem to get (N,v^P) can be stated as:

$$\begin{cases} \min\{\|v^* - v\|^2 \\ s.t \ \left\langle v^*,\delta^{\mathcal{B}}\right\rangle \leq 0 \text{ for all } \mathcal{B} \in \mathcal{M}(N). \end{cases} \tag{1}$$

From the Karush-Kuhn-Tucker necessary first order conditions for the problem (1), we can get some insight about the game (N,v^P) (see Cesco [5]).

Theorem 1. *Let a non-balanced game (N,v) be given. Then, there is $\mathcal{B}^* \in \mathcal{B}(N)$, a set of balancing weights $\lambda^{\mathcal{B}^*}$ for $\mathcal{B}^*$, and a positive number ε such that $v^P = v - \varepsilon\delta^{\mathcal{B}^*,\lambda^{\mathcal{B}^*}}$.*

[3] Here we are identifying each game (N,v) with its characteristic function v. Moreover, since always $v(\phi) = 0$, we can describe this function by a (2^n-1)-dimensional vector.

[4] The $(n-2)$-dimensional face determined by $\delta^{\mathcal{B}}$ is the set of all games (N,v) satisfying $\left\langle v,\delta^{\mathcal{B}}\right\rangle = 0$, and $\left\langle v,\delta^{\mathcal{B}^*}\right\rangle \leq 0$ for any other $\mathcal{B}^* \in \mathcal{M}(N)$.

The explicit expression for v^P is worth of further analysis. To this end, given a game (N, v), a family $\mathcal{B} \in \mathcal{B}(N)$ with balancing weights $\lambda^{\mathcal{B}}$, and $\varepsilon > 0$, let $(N, v^{\mathcal{B},\lambda^{\mathcal{B}},\varepsilon})$ be the game whose characteristic function is $v^{\mathcal{B},\lambda^{\mathcal{B}},\varepsilon} = v - \varepsilon\delta^{\mathcal{B},\lambda^{\mathcal{B}}}$. Clearly, $v^{\mathcal{B},\lambda^{\mathcal{B}},\varepsilon}(S) = v(S) - \varepsilon\lambda^{\mathcal{B}}_S$ if $S \in \mathcal{B}, v^{\mathcal{B},\lambda^{\mathcal{B}},\varepsilon}(N) = v(N) + \varepsilon$, and $v^{\mathcal{B},\lambda^{\mathcal{B}},\varepsilon}(S) = v(S)$ otherwise. Thus, in $(N, v^{\mathcal{B},\lambda^{\mathcal{B}},\varepsilon})$, show up simultaneously the two basic effects observed separately in (N, v^{ε}) and $(N, v^{\mathcal{B}})$, the auxiliary balanced games used to define the ε-core and the aspiration core respectively. Some of the intermediate coalitions, namely, those coalitions in $\mathcal{B}$, decrease their value, while the grand coalition N increases its value. Of course, this behavior is shared by (N, v^P). Whenever $\mathcal{B} \in \mathcal{M}(N)$ we will use the simpler notation $v^{\mathcal{B},\varepsilon}$ for $v^{\mathcal{B},\lambda^{\mathcal{B}},\varepsilon}$.

The set of the active constraints $\{\mathcal{B} \in \mathcal{M}(N) : \langle v^P, \delta^{\mathcal{B}} \rangle = 0\}$ at the solution v^P of (1) is related, somehow, to those objecting families in $\mathcal{B}^w(N, v)$, as it is shown in a corollary of the next result, whose proof can be found in Cesco [5]).

Lemma 1. *Let (N, v) be a game a non-balanced game, and $\mathcal{B} \in \mathcal{B}(N)$ be such that $\Delta(\mathcal{B}, \lambda^{\mathcal{B}}, v) > 0$ for some set $\lambda^{\mathcal{B}}$ of balancing weights. Then, for any other $\mathcal{B}^* \in \mathcal{B}(N)$, for any $\varepsilon > 0$,*

$$\Delta(\mathcal{B}^*, \lambda^{\mathcal{B}^*}, v^{\mathcal{B},\varepsilon}) = \Delta(\mathcal{B}^*, \lambda^{\mathcal{B}^*}, v) - \varepsilon(1 + \sum_{S\in\mathcal{B}^*\cap\mathcal{B}} \lambda^{\mathcal{B}^*}_S \lambda^{\mathcal{B}}_S).$$

Corollary 1. *Let (N, v) be a game a non-balanced game, and $\mathcal{B} \in \mathcal{B}(N)$ be such that $\Delta(\mathcal{B}, \lambda^{\mathcal{B}}, v) > 0$ for some set $\lambda^{\mathcal{B}}$ of balancing weights. Then, for any $\varepsilon > 0$, if $\mathcal{B}^* \in \mathcal{B}(N)$ is such that $\Delta(\mathcal{B}^*, \lambda^{\mathcal{B}^*}, v) \leq 0$, $\Delta(\mathcal{B}^*, \lambda^{\mathcal{B}^*}, v^{\mathcal{B},\varepsilon}) < 0$.*

Corollary 2. *Let (N, v) be a non-balanced game. If $v^P = v - \varepsilon\delta^{\mathcal{B},\lambda^{\mathcal{B}}}$ for some $\mathcal{B} \in \mathcal{B}(N)$, a set of balancing weights $\lambda^{\mathcal{B}}$ for $\mathcal{B}$, and a positive number ε, then $\{\mathcal{B}^* \in \mathcal{M}(N) : \langle v^P, \delta^{\mathcal{B}^*} \rangle = 0\} \subseteq \mathcal{B}^w(N, v)$.*

3 Obtention of (N, v^P) in some particular cases

In some cases, the optimization problem (1) can be solved in a simple way, by taking advantage of the characterization for the $(n-2)$-dimensional faces of $\mathbb{V}^n$. In fact, for any $\mathcal{B} \in \mathcal{M}(N)$, and when ε is chosen in an appropriate way, $v^{\mathcal{B},\varepsilon}(S) = v - \varepsilon\delta^{\mathcal{B}}$ is the orthogonal projection of a non-balanced game (N, v) onto the $(n-2)$-dimensional subspace containing the face of $\mathbb{V}^n$ determined by $\mathcal{B}$. The following result concerns with this projection procedure and will be useful in what follows.

Lemma 2. *Let (N, v) be a non-balanced game, and $\mathcal{B} \in \mathcal{B}(N)$ be such that $\Delta(\mathcal{B}, \lambda^{\mathcal{B}}, v) > 0$ for some set $\lambda^{\mathcal{B}}$ of balancing weights. If*

$$\varepsilon = \frac{\Delta(\mathcal{B}, \lambda^{\mathcal{B}}, v)}{1 + \sum\limits_{S\in\mathcal{B}} (\lambda^{\mathcal{B}}_S)^2} \tag{2}$$

then, $w(\mathcal{B}, \lambda^{\mathcal{B}}, v^{\mathcal{B},\lambda^{\mathcal{B}},\varepsilon}) = v^{\mathcal{B},\lambda^{\mathcal{B}},\varepsilon}(N)$.

Proof. Clearly $\varepsilon > 0$.

$$\begin{aligned} w(\mathcal{B}, \lambda^{\mathcal{B}}, v^{\mathcal{B},\lambda^{\mathcal{B}},\varepsilon}) &= \sum_{S\in\mathcal{B}} \lambda_S^{\mathcal{B}} v^{\mathcal{B},\lambda^{\mathcal{B}}\varepsilon}(S) \\ &= \sum_{S\in\mathcal{B}} \lambda_S^{\mathcal{B}}(v(S) - \varepsilon\lambda_S^{\mathcal{B}}) \\ &= w(N,v) - v(N) + v(N) - \varepsilon \sum_{S\in\mathcal{B}} (\lambda_S^{\mathcal{B}})^2 \\ &= v(N) + \Delta(\mathcal{B}, \lambda^{\mathcal{B}}, v) - \frac{\Delta(\mathcal{B}, \lambda^{\mathcal{B}}, v)}{1 + \sum_{S\in\mathcal{B}} (\lambda_S^{\mathcal{B}})^2} \sum_{S\in\mathcal{B}} (\lambda_S^{\mathcal{B}})^2 \\ &= v(N) + \frac{\Delta(\mathcal{B}, \lambda^{\mathcal{B}}, v)}{1 + \sum_{S\in\mathcal{B}} (\lambda_S^{\mathcal{B}})^2} = v^{\mathcal{B},\lambda^{\mathcal{B}},\varepsilon}(N). \end{aligned}$$

The following result gives a simple solution of the problem (1) when a coalition exhibits some kind of "dominant" position (see Cesco [5]).

Proposition 1. *Let (N, v) be a game a non-balanced game, $\mathcal{B} \in \mathcal{B}^w(N, v)$ and ε given by* (2). *If for any other $\mathcal{B}^* \in \mathcal{M}(N), w(\mathcal{B}^*, \lambda^{\mathcal{B}^*}, v) - v(N) \leq \varepsilon(1 + \sum_{S\in\mathcal{B}^*\cap\mathcal{B}} \lambda_S^{\mathcal{B}^*} \lambda_S^{\mathcal{B}})$, then $(N, v^{\mathcal{B},\varepsilon})$ is a balanced game belonging to the $(n-2)$-dimensional face determined by $\delta^{\mathcal{B}}$.*

A particular simple case of Proposition 1 is when $\mathcal{B}^w(N, v)$ contains only one element.

Corollary 3. *Let (N, v) be a game, and such that $\mathcal{B}^w(N, v) = \{\mathcal{B}\}$. If ε is given by (2), the game $(N, v^{\mathcal{B},\varepsilon})$ coincides with (N, v^P).*

4 Other approaches to compute v^P

In this section we deal with a case where the condition stated in Proposition 1 is not satisfied. We think that the approaches we use to solve it could be useful to learn how to tackle the general case. We propose two alternative methods to obtain v^P. The first one is an algebraic method while the second is a geometric one. We start with the following auxiliary result.

Lemma 3. *Let (N, v) be a game, and $\mathcal{B} \in \mathcal{B}^w(N, v)$ be such that $w(\mathcal{B}, \lambda^{\mathcal{B}}, v) = w(N, v)$. Also, let $\mathcal{B}^* \in \mathcal{B}^w(N, v)$ with $w(\mathcal{B}^*, \lambda^{\mathcal{B}^*} v) - v(N) > \varepsilon(1 + \sum_{S\in\mathcal{B}^*\cap\mathcal{B}} \lambda_S^{\mathcal{B}^*} \lambda_S^{\mathcal{B}})$, where ε is given by* (2). *Then,*

$$\sum_{S\in\mathcal{B}} (\lambda_S^{\mathcal{B}})^2 > \sum_{S\in\mathcal{B}^*\cap\mathcal{B}} \lambda_S^{\mathcal{B}^*} \lambda_S^{\mathcal{B}}.$$

Proof. Taking into account the definition of ε, clearly the balancing weights of $\mathcal{B}^*$ satisfy the following inequalities:

$$\Delta(N,v)\frac{1+\sum\limits_{S\in\mathcal{B}^*\cap\mathcal{B}}\lambda_S^{\mathcal{B}^*}\lambda_S^{\mathcal{B}}}{1+\sum\limits_{S\in\mathcal{B}}(\lambda_S^{\mathcal{B}})^2} < w(\mathcal{B}^*,\lambda^{\mathcal{B}^*},v)-v(N)$$
$$\leq \Delta(N,v).$$

Therefore, it must hold that

$$\frac{1+\sum\limits_{S\in\mathcal{B}^*\cap\mathcal{B}}\lambda_S^{\mathcal{B}^*}\lambda_S^{\mathcal{B}}}{1+\sum\limits_{S\in\mathcal{B}}(\lambda_S^{\mathcal{B}})^2} < 1. \tag{3}$$

But, from (3) we easily get that

$$\sum_{S\in\mathcal{B}}(\lambda_S^{\mathcal{B}})^2 > \sum_{S\in\mathcal{B}^*\cap\mathcal{B}}\lambda_S^{\mathcal{B}^*}\lambda_S^{\mathcal{B}}.$$

The preceding lemma allows us to extend, somehow, Corollary 3. In what follows, we will consider $\lambda^{\mathcal{B}^1}$ (and similarly $\lambda^{\mathcal{B}^2}$) as a (2^n-1)-dimensional vector whose entries are indexed by the coalitions in $\mathcal{P}(N)$. Moreover, the value of any entry indexd by a coalition $S\in\mathcal{B}^1$ will be $\lambda_S^{\mathcal{B}^1}$, and 0 otherwise. With this convention in mind, $\left\|\lambda^{\mathcal{B}}\right\|^2=\sum\limits_{S\in\mathcal{B}}(\lambda_S^{\mathcal{B}})^2$ and $\left\langle\lambda_S^{\mathcal{B}^1},\lambda_S^{\mathcal{B}^2}\right\rangle=\sum\limits_{S\in\mathcal{B}^1\cap\mathcal{B}^2}\lambda_S^{\mathcal{B}^1}\lambda_S^{\mathcal{B}^2}$.

Theorem 2. *Let (N,v) be a game. If $\mathcal{B}^w(N,v)=\{\mathcal{B}^1,\mathcal{B}^2\}$, and $\mathcal{B}^1$ is such $w(\mathcal{B}^1,v)=w(N,v)>w(\mathcal{B}^2,v)$. If*

1) $w(\mathcal{B}^2,\lambda^{\mathcal{B}^2},v)-v(N)>\varepsilon^1(1+\sum\limits_{S\in\mathcal{B}^1\cap\mathcal{B}^2}\lambda_S^{\mathcal{B}^1}\lambda_S^{\mathcal{B}^2})$, with $\varepsilon^1=\frac{\Delta(\mathcal{B}^1,v)}{1+\sum\limits_{S\in\mathcal{B}^1}(\lambda_S^{\mathcal{B}^1})^2}$,

and

2) $w(\mathcal{B}^1,\lambda^{\mathcal{B}^1},v)-v(N)>\varepsilon^2(1+\sum\limits_{S\in\mathcal{B}^1\cap\mathcal{B}^2}\lambda_S^{\mathcal{B}^1}\lambda_S^{\mathcal{B}^2})$, with $\varepsilon^2=\frac{\Delta(\mathcal{B}^2,v)}{1+\sum\limits_{S\in\mathcal{B}^1}(\lambda_S^{\mathcal{B}^2})^2}$

then $v^P=v-\varepsilon\sum\limits_{i=\{1,2\}}\frac{\varepsilon_i}{\varepsilon_1+\varepsilon_2}\delta^{\mathcal{B}^i}$, where ε_1 and ε_2 and $\varepsilon_0=\frac{1}{\varepsilon}$ are positive numbers satisfying the linear system

$$\begin{cases}\Delta(\mathcal{B}^1,v)\varepsilon_0-\left\|\lambda^{\mathcal{B}^1}\right\|^2\varepsilon_1-\left\langle\lambda^{\mathcal{B}^1},\lambda^{\mathcal{B}^2}\right\rangle\varepsilon_2=1\\ \Delta(\mathcal{B}^2,v)\varepsilon_0-\left\langle\lambda^{\mathcal{B}^1},\lambda^{\mathcal{B}^2}\right\rangle\varepsilon_1-\left\|\lambda^{\mathcal{B}^2}\right\|^2\varepsilon_2=1\\ \varepsilon_1+\varepsilon_2=1,\end{cases} \tag{4}$$

Proof. During the proof, Δ_1 will stand for $\Delta(\mathcal{B}^1,v)$, and Δ_2 will stand for $\Delta(\mathcal{B}^2,v)$. We claim that the linear system (4) has only one solution. Indeed, the discriminant of (4) is

$$\Delta=\Delta_1(\left\|\lambda^{\mathcal{B}^2}\right\|^2-\left\langle\lambda^{\mathcal{B}^1},\lambda^{\mathcal{B}^2}\right\rangle)+\Delta_2(\left\|\lambda^{\mathcal{B}^1}\right\|^2-\left\langle\lambda^{\mathcal{B}^1},\lambda^{\mathcal{B}^2}\right\rangle).$$

If $\Delta \leq 0$, we would get that

$$-\Delta_2(\left\|\lambda^{\mathcal{B}^1}\right\|^2 - \left\langle\lambda^{\mathcal{B}^1}, \lambda^{\mathcal{B}^2}\right\rangle) = -\Delta_2(1 + \left\|\lambda^{\mathcal{B}^1}\right\|^2 - (1 + \left\langle\lambda^{\mathcal{B}^1}, \lambda^{\mathcal{B}^2}\right\rangle))$$
$$\geq \Delta_1(1 + \left\|\lambda^{\mathcal{B}^2}\right\|^2 - (1 + \left\langle\lambda^{\mathcal{B}^1}, \lambda^{\mathcal{B}^2}\right\rangle)).$$

But now, by using the hypotesis 1) and 2), we would also get that

$$(\Delta_1 + \Delta_2)(1 + \left\langle\lambda^{\mathcal{B}^1}, \lambda^{\mathcal{B}^2}\right\rangle) \geq \Delta_1(1 + \left\|\lambda^{\mathcal{B}^2}\right\|^2) + \Delta_2(1 + \left\|\lambda^{\mathcal{B}^1}\right\|^2)$$
$$> \Delta_2(1 + \left\langle\lambda^{\mathcal{B}^1}, \lambda^{\mathcal{B}^2}\right\rangle) + \Delta_1(1 + \left\langle\lambda^{\mathcal{B}^1}, \lambda^{\mathcal{B}^2}\right\rangle)$$
$$= (\Delta_1 + \Delta_2)(1 + \left\langle\lambda^{\mathcal{B}^1}, \lambda^{\mathcal{B}^2}\right\rangle),$$

a contradiction. Therefore, $\Delta > 0$, and the unique solution of (4) will allow us to compute the numbers $\varepsilon_1, \varepsilon_2$ and $\varepsilon = \frac{1}{\varepsilon_0}$. We will show that this solution is a positive one.

By Cramer's rule, $\varepsilon_0 = \frac{\Delta_{\varepsilon_0}}{\Delta}$, where

$$\Delta_{\varepsilon_0} = \left\|\lambda^{\mathcal{B}^1}\right\|^2 + \left\|\lambda^{\mathcal{B}^2}\right\|^2 - 2\left\langle\lambda^{\mathcal{B}^1}, \lambda^{\mathcal{B}^2}\right\rangle + \left\|\lambda^{\mathcal{B}^1}\right\|^2 \left\|\lambda^{\mathcal{B}^2}\right\|^2 - \left\langle\lambda^{\mathcal{B}^1}, \lambda^{\mathcal{B}^2}\right\rangle^2$$
$$= (\left\|\lambda^{\mathcal{B}^1}\right\| - \|\lambda^{\mathcal{B}2}\|)^2 + \left\|\lambda^{\mathcal{B}^1}\right\|^2 \left\|\lambda^{\mathcal{B}^2}\right\|^2 - \left\langle\lambda^{\mathcal{B}^1}, \lambda^{\mathcal{B}^2}\right\rangle^2,$$

which is also a positive number because of Hölder's inequality and the fact that $\lambda^{\mathcal{B}^1}$ and $\lambda^{\mathcal{B}^2}$ are linearly independent vectors . Then, ε_0 turns to be a positive number, and so it is $\varepsilon = \frac{1}{\varepsilon_0}$.

Also by Cramer's rule, $\varepsilon_1 = \frac{\Delta_{\varepsilon_1}}{\Delta}$ and $\varepsilon_2 = \frac{\Delta_{\varepsilon_2}}{\Delta}$. But

$$\Delta_{\varepsilon_1} = (\Delta_1 - A_2) + \Delta_1 \left\|\lambda^{\mathcal{B}^1}\right\|^2 - \Delta_2 \left\langle\lambda^{\mathcal{B}^1}, \lambda^{\mathcal{B}^2}\right\rangle$$
$$(\Delta_1 - A_2) + \Delta_2(\left\|\lambda^{\mathcal{B}^1}\right\|^2 - \left\langle\lambda^{\mathcal{B}^1}, \lambda^{\mathcal{B}^2}\right\rangle)$$
$$> 0$$

since $\Delta_1 - A_2 > 0$ and $\left\|\lambda^{\mathcal{B}^1}\right\|^2 - \left\langle\lambda^{\mathcal{B}^1}, \lambda^{\mathcal{B}^2}\right\rangle \geq 0$ because of Lemma 3.

On the other hand,

$$\Delta_{\varepsilon_2} = -\Delta_1(1 + \left\langle\lambda^{\mathcal{B}^1}, \lambda^{\mathcal{B}^2}\right\rangle) + A_2(1 + \left\|\lambda^{\mathcal{B}^1}\right\|^2)$$

is also a positive number because of condition 1). Therefore, $\varepsilon, \varepsilon_1$ and ε_2 are positive numbers.

Now, we are going to prove that (N, v^P) coincides with $(N, v^{\mathcal{B},\lambda^{\mathcal{B}},\varepsilon})$. We point out that $(N, v^{\mathcal{B},\lambda^{\mathcal{B}},\varepsilon}) \in \mathbb{V}^n$. In fact, since $\varepsilon_1, \varepsilon_2$ and $\varepsilon_0 = \frac{1}{\varepsilon}$ satisfy system (4), we get that

$$\Delta(\mathcal{B}^1, v) - \varepsilon(\varepsilon_1 \left\|\lambda^{\mathcal{B}^1}\right\|^2 + \left\langle\lambda^{\mathcal{B}^1}, \lambda^{\mathcal{B}^2}\right\rangle \varepsilon_2) = \varepsilon.$$

But now, taking into account that

$$w(\mathcal{B}^1, \lambda^{\mathcal{B}^1}, v^{\mathcal{B},\lambda^{\mathcal{B}},\varepsilon}) = w(\mathcal{B}^1, v) - \varepsilon(\varepsilon_1 \sum_{S\in\mathcal{B}^1} (\lambda_S^{\mathcal{B}^1})^2 + \varepsilon_2 \sum_{S\in\mathcal{B}^1\cap\mathcal{B}^2} \lambda_S^{\mathcal{B}^1} \lambda_S^{\mathcal{B}^2}),$$

and that $\Delta(\mathcal{B}^1, v) = w(\mathcal{B}^1, v) - v(N)$ we get that

$$\begin{aligned} w(\mathcal{B}^1, \lambda^{\mathcal{B}^1}, v^{\mathcal{B},\lambda^{\mathcal{B}},\varepsilon}) &= v(N) + \varepsilon \\ &= v^{\mathcal{B},\lambda^{\mathcal{B}},\varepsilon} \end{aligned}$$

Similarly, we get that

$$w(\mathcal{B}^1, \lambda^{\mathcal{B}^1}, v^{\mathcal{B},\lambda^{\mathcal{B}},\varepsilon}) = v^{\mathcal{B},\lambda^{\mathcal{B}},\varepsilon}$$

too. On the other hand, because of Corollary 1, we have that $w(\mathcal{B}^*, \lambda^{\mathcal{B}^*}, v^{\mathcal{B},\lambda^{\mathcal{B}},\varepsilon}) \leq v^{\mathcal{B},\lambda^{\mathcal{B}},\varepsilon}$ and for any other $\mathcal{B}^* \in \mathcal{M}(N)$, different from $\mathcal{B}^1$ and $\mathcal{B}^2$. Thus, our claim follows from the Shapley-Bondareva Theorem. Finally, to show that $(N, v^{\mathcal{B},\lambda^{\mathcal{B}},\varepsilon}) = (N, v^P)$, we note that, for any $(N, v^*) \in \mathbb{V}^n$,

$$\begin{aligned} \left\langle v - v^{\mathcal{B},\lambda^{\mathcal{B}},\varepsilon}, v^* - v^{\mathcal{B},\lambda^{\mathcal{B}},\varepsilon} \right\rangle &= \left\langle \varepsilon\delta^{\mathcal{B},\lambda^{\mathcal{B}}}, v^* - v^{\mathcal{B},\lambda^{\mathcal{B}},\varepsilon} \right\rangle \\ &= \left\langle \varepsilon\delta^{\mathcal{B},\lambda^{\mathcal{B}}}, v^* \right\rangle \\ &= \varepsilon(\varepsilon_1 \left\langle \delta^{\mathcal{B}^1}, v^* \right\rangle + \varepsilon_2 \left\langle \delta^{\mathcal{B}^1}, v^* \right\rangle) \\ &\leq 0, \end{aligned}$$

which is a condition that guarantees that $(N, v^{\mathcal{B},\lambda^{\mathcal{B}},\varepsilon})$ is the closest game in $\mathbb{V}^n$ to the given non-balanced game (N, v).

Remark 2. Conditions 1) and 2) in Theorem 2 rule out the cases in which (N, v^P) is the projection of (N, v) onto the faces determined by $\delta^{\mathcal{B}^1}$ or $\delta^{\mathcal{B}^2}$, in which cases (N, v^P) is obtained in a more direct way.

We now turn our attention to the second method. We start with the non-balanced game (N, v), and we assume, as before, that $\mathcal{B}^w(N, v) = \{\mathcal{B}^1, \mathcal{B}^2\}$. Then we construct the game $(N, v^{\mathcal{B}^1,\varepsilon_1})$ as the projection of (N, v) onto $\mathbb{V}^n$ following the direction of $\delta^{\mathcal{B}^1}$. Let us call this game (N, v^1). Clearly this game satisfies $\Delta(\mathcal{B}^1, v^1) = 0$. If $\Delta(\mathcal{B}^2, v^1) \leq 0$, then $(N, v^1) = (N, v^P)$ and we are done. If this is not the case, $\Delta(\mathcal{B}^2, v^1) > 0$ and we construct the game $(N, v^{1\mathcal{B}^2,\varepsilon_2})$ as the projection of (N, v^1) onto $\mathbb{V}^n$ following the direction of $\delta^{\mathcal{B}^2}$. Let us call this new game (N, v^2). It satisfies that $\Delta(\mathcal{B}^2, v^2) = 0$, and $\Delta(\mathcal{B}^1, v^2) < 0$.

We now repeat the process, also starting from (N, v) but now, reversing the order of the projections. We first construct the game (N, v_2), and if $(N, v_2) \neq (N, v^P)$, we then construct the game (N, v_1). In general, $(N, v_1) \neq (N, v^2)$.

Clearly (N, v_2) satisfies $\Delta(\mathcal{B}^2, v_2) = 0$, and $\Delta(\mathcal{B}^1, v_2) > 0$ and (N, v_1) satisfies $\Delta(\mathcal{B}^1, v_1) = 0$, and $\Delta(\mathcal{B}^2, v_1) < 0$.

We now construct the game (N, v^μ) with $v^\mu = \mu v^1 + (1-\mu)v_1$. Since

$$\begin{aligned} w(\mathcal{B}^2, v^\mu) &= \sum_{S\in\mathcal{B}^2} \lambda^{\mathcal{B}^2} v^\mu(S) \\ &= \sum_{S\in\mathcal{B}^2} \lambda^{\mathcal{B}^2}(\mu v^1(S) + (1-\mu)v_1(S)) \\ &= \mu w(\mathcal{B}^2, v^1) + (1-\mu)w(\mathcal{B}^2, v_1), \end{aligned}$$

and

$$v^\mu(N) = \mu v^1(N) + (1-\mu)v_1(N),$$

we conclude that

$$\Delta(\mathcal{B}^2, v^\mu) = \mu\Delta(\mathcal{B}^2, v^1) + (1-\mu)\Delta(\mathcal{B}^2, v_1). \tag{5}$$

Similarly,

$$\Delta(\mathcal{B}^1, v^\mu) = \mu\Delta(\mathcal{B}^1, v^1) + (1-\mu)\Delta(\mathcal{B}^1, v_1). \tag{6}$$

From (6), and the fact that $\Delta(\mathcal{B}^1, v^1) = \Delta(\mathcal{B}^1, v_1) = 0$, we get that, for any $\mu, \Delta(\mathcal{B}^1, v^\mu) = 0$. On the other hand, since $\Delta(\mathcal{B}^2, v^1) > 0$, and $\Delta(\mathcal{B}^2, v_1) < 0$, there is a unique $0 < \mu^* < 1$ for which $\Delta(\mathcal{B}^2, v^\mu) = 0$ too. We claim that $(N, v^{\mu^*}) = (N, v^P)$. To see this, we first note that

$$\begin{aligned} v^{\mu^*} &= \mu^*(v - \varepsilon^1\delta^{\mathcal{B}^1}) + (1-\mu^*)(v - \varepsilon_2\delta^{\mathcal{B}^2} - \varepsilon_1\delta^{\mathcal{B}^1}) \\ &= v - (\mu^*\varepsilon^1 + (1-\mu^*)\varepsilon_1)\delta^{\mathcal{B}^1} + (1-\mu^*)\varepsilon_2\delta^{\mathcal{B}^2}). \end{aligned}$$

Since both $\mu^*\varepsilon^1 + (1-\mu^*)\varepsilon_1$ and $(1-\mu^*)\varepsilon_2$ turn to be positive numbers, we can put that

$$\begin{aligned} v^{\mu^*} &= v - \varepsilon^*\delta^{\mathcal{B},\lambda^{\mathcal{B}}} \\ &= v^{\mathcal{B},\lambda^{\mathcal{B}},\varepsilon^*}, \end{aligned}$$

where $\mathcal{B} = \mathcal{B}^1 \cup \mathcal{B}^2$,

$$\varepsilon^* = (\mu^*\varepsilon^1 + (1-\mu^*)\varepsilon_1) + (1-\mu^*)\varepsilon_2,$$

and

$$\lambda^{\mathcal{B}} = \frac{(\mu^*\varepsilon^1 + (1-\mu^*)\varepsilon_1)}{\varepsilon^*}\lambda^{\mathcal{B}^1} + \frac{(1-\mu^*)\varepsilon_2)}{\varepsilon^*}\lambda^{\mathcal{B}^2}.$$

Therefore, according to Corollary 1 (note that $\Delta(\mathcal{B}, \lambda^{\mathcal{B}}, v) > 0$), $\Delta(\mathcal{B}^*, v^{\mathcal{B},\lambda^{\mathcal{B}},\varepsilon^*}) < 0$ for any other $\mathcal{B}^* \in \mathcal{M}(N), \mathcal{B}^* \neq \mathcal{B}^1, \mathcal{B}^2$. This proves that $(N, v^{\mathcal{B},\lambda^{\mathcal{B}},\varepsilon^*})$ belongs to $\mathbb{V}^n$, more precisely, to the face determined by $\delta^{\mathcal{B}^1}$ and $\delta^{\mathcal{B}^1}$. Moreover, since as before, for any $v^* \in \mathbb{V}^n$, we get that

$$\begin{aligned} \left\langle v - v^{\mathcal{B},\lambda^{\mathcal{B}},\varepsilon^*}, v^* - v^{\mathcal{B},\lambda^{\mathcal{B}},\varepsilon^*} \right\rangle &= \left\langle \varepsilon^*\delta^{\mathcal{B},\lambda^{\mathcal{B}}}, v^* - v^{\mathcal{B},\lambda^{\mathcal{B}},\varepsilon^*} \right\rangle \\ &\leq 0, \end{aligned}$$

we have that $(N, v^{\mathcal{B},\lambda^{\mathcal{B}},\varepsilon^*}) = (N, v^P)$.

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5 Final remarks

The linear system first (4) related to the fist method presented in Section 3 can be easily genealized to the case that $\mathcal{B}^w(N,v) = \{\mathcal{B}^1, ..., \mathcal{B}^k\}$. However, we do no have yet a general condition guaranteeing the non-negativity of its solution. However, in the case that we were able to get, in advance, the family $\{\mathcal{B}^* \in \mathcal{M}(N) : \langle v^P, \delta^{\mathcal{B}^*} \rangle = 0\}$ related to the active constraints at v^P, the existence and the non-negativity of the solution could be assured by the Kruskal-Kuhn-Tucker conditions of (1). Thus, our attention now is focused in developing a practical procedure to find the set $\{\mathcal{B}^* \in \mathcal{M}(N) : \langle v^P, \delta^{\mathcal{B}^*} \rangle = 0\}$. The second method presented in Section 3 provides, in the particular case worked there, a way to find the required set. To this end, we have to consider successive orthogonal projections of the type indicated in Lemma 2. The idea behind this particular case motivate us to propose a general algorithmic version, in the form of a pseudo-code. At each step, only one new objecting family is brought into consideration, and the associated orthogonal projection is obtained. The final result for any given game (N,v), is $(N,\bar{v}^P)$. We call this procedure, the P-procedure which is defined in the following pseudo-code.

Data: A game (N,v).
Result: $(N,\bar{v}^P)$.

1. Let $(N,v^*) = (N,v)$ and $\mathcal{LB} = \phi$
2. Is $(N,v^*) \in \mathbb{V}^n$? If YES, $(N,v^*) = (N,\bar{v}^P)$ and go to end. Else find an objecting minimal balanced family $\mathcal{B}$ and let $\mathcal{LB} = \mathcal{LB} \cup \{\mathcal{B}\}$.
3. Let $(N,v^*) = (N, v^{*\mathcal{B},\lambda^{\mathcal{B}},\varepsilon^{\mathcal{B}}})$ with $\varepsilon^{\mathcal{B}} = \frac{\Delta(\mathcal{B},v^*)}{1+\sum_{S\in\mathcal{B}}(\lambda_S^{\mathcal{B}})^2}$.
4. Is $(N,v^*) \in \mathbb{V}^n$? If YES, $(N,v^*) = (N,\bar{v}^P)$ and go to end. Else find an objecting minimal balanced family $\mathcal{B}$ and let $\mathcal{LB} = \mathcal{LB} \cup \{\mathcal{B}\}$.
5. Go to Step 3.

Steps 3 and 4 are the key steps in the P-procedure. We now give the way we practically solve Step 4. We mention that at each Step 4, until the end, $\mathcal{LB}$ always incorporates a new minimal balanced family, different from all the others taken into account in the previous steps. This will guarantee the ending of the algorithm in a finite number of steps.

5.1 About Step 4

To go round Step 4 we use an algorithm developed by Cesco [3], originally designed to compute a core-imputation of a balanced game (N,v). However, if the game is a non-balanced one, the final result of the algorithm is a finite sequence $((x^i,S^i))_{i=1}^{m+1}$, where, for each i, x^i is a pre-imputation and S^i is a coalition with $e(S^i,x^i) = 0$. Moreover, $S^{m+1} = S^1$, and $\left\|x^{m+1} - x^1\right\|_2^2 \leq tol$, where *tol* stands for the machine tolerance number. Namely, the final result of the algorithm is almost a cycle. We conjecture that it should "converge" to a true cycle, but we have been able to prove this fact only in some particular cases. However, the interesting point is that the family of coalitions $(S^i)_{i=1}^m$ is a balanced one, and a set of balancing weights λ for it can be computed very closely in terms of the family of pre-imputations $(x^i)_{i=1}^m$ (Cesco [4]). Two other important properties have shown up during several simulation exercises, although we do not have formal proof, except for some particular situations. First, the family of coalitions has maximal worth (within the limits that the machine precision imposes). Second, it is a minimal balanced family. If this were not the case, an additional procedure can extract a minimal subfamily. And if this minimal family does not have maximal worth, a finite sequence of applications of the aforementioned two procedures will lead to a maximal worth balanced family of coalitions. It is worth mentioning that we have theoretic results supporting the use of approximate cycles instead of true cycles (Cesco [6]).

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