MiniMax Affine Estimation of Parameters of Multiple Damped Complex Exponentials

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Abstract. Multiple damped complex exponentials are of great practical importance as they are useful for describing many technological situations. Several estimators have been developed for the parameters of these complex exponentials. In this paper, we apply the MiniMax affine estimator to this problem in order to obtain a better performance (in terms of the mean squared error) than other unbiased estimators. Through simulations, this estimator is shown to have a reduced mean squared error, especially for the adverse case of lower signal-to-noise ratio. Additionally, a closed form expression for the MiniMax affine estimator is presented.

Keywords: Damped Sinusoids Estimation, Affine Estimators, Mean Squared Error

1 Introduction

Multiple Damped Complex Exponentials are used to describe a myriad of technological applications. In particular, they arise in problems related to linear system identification [1,2], speech analysis [3], analysis of data obtained through a nuclear magnetic resonance [4] and photoacoustic beam profiling of pulsed lasers [5].

The general model will be described as

$$R(k) = \sum_{i=1}^{q} x_i(k) + N(k) \tag{1}$$

where

$$x_i(k) = c_i \mathrm{e}^{j\,\beta_i}\, \mathrm{e}^{(\alpha_i + j\,\omega_i)k} \tag{2}$$

with $\alpha_i < 0$, $c_i > 0$, $\omega_i \in [0, 2\pi)$, $\beta_i \in [0, 2\pi)$ for every $i = 1, \ldots, q$ and $j^2 = -1$. Also, $\{N(k)\}$ is a discrete-time, circularly-symmetric [6], zero-mean, complex additive white gaussian noise (AWGN), with $\mathbb{E}[N(k)N^*(k+m)] = \sigma^2\delta(m)$, with σ^2 known. The SNR is defined as $\mathrm{SNR} = \sum_{i=1}^{q} c_i^2/\sigma^2$. A sample of size n is available, $\{R(k)\}_{k=0}^{n-1}$, and the deterministic parameter vector to be estimated is

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$\boldsymbol{\theta} = \left[\boldsymbol{\theta}_1^T \ldots \boldsymbol{\theta}_q^T\right]^T \in \mathbb{R}^{4q}$ where $\boldsymbol{\theta}_i = \left[\alpha_i \; c_i \; \omega_i \; \beta_i\right]^T \in \mathbb{R}^4$, $i = 1, \ldots, q$. Furthermore, it is assumed that the number of damped complex exponentials is known. Otherwise, there are different methods for determining the value of q [7,8].

Many different approaches have been adopted to estimate the parameter $\boldsymbol{\theta}$ of model (1) [9,10,11,12,13]. In particular, the attention will be focused on the estimator developed by Umesh and Tufts [4]. This estimator is called the Fast Maximum Likelihood (FML) estimator and is Cramér-Rao efficient for moderately high values of SNR ([4], p. 2249).

The FML estimator is an unbiased estimator. However, it was shown that in many situations there are biased estimators that have a lower **MSE** than the unbiased ones [14]. Hence, work has been carried out to obtain better biased estimators [15,16,17]. One particular idea for these transformations, developed by Y. C. Eldar [18,19,20], consists in obtaining a biased estimator through an affine transformation of an unbiased one.

Let $\mathbf{h}\{R(k)\}$ be the unbiased FML estimator [4] for the parameter $\boldsymbol{\theta} \in \mathbb{R}^{4q}$ based on sample $\{R(k)\}_{k=0}^{n-1}$, then an affine estimator can be obtained,

$$\mathbf{h}_B\{R(k)\} = \mathbf{A}\,\mathbf{h}\{R(k)\} + \mathbf{b} \tag{3}$$

where $\mathbf{A} \in \mathbb{R}^{4q\times 4q}$ and $\mathbf{b} \in \mathbb{R}^{4q}$. It is evident that the affine estimator $\mathbf{h}_B\{R(k)\}$ is biased and that its bias is an affine function of the paramater $\boldsymbol{\theta}$. Additionally, the parameter is supposed to belong to a known bounded region $\mathcal{V} \subseteq \mathbb{R}^{4q}$ that is called *validation-region*. This restriction is particularly useful for problems where the parameter is bounded or there exists some previous information about where it lies.

The objective of any affine estimator, then, is to find the values of $\mathbf{A}$ and $\mathbf{b}$ that make the biased estimator dominant, this is

$$\mathbf{MSE}(\mathbf{h}_B\{R(k)\}) \leq \mathbf{MSE}(\mathbf{h}\{R(k)\}) \quad , \quad \forall \boldsymbol{\theta} \in \mathcal{V} \tag{4}$$

with strict inequality for at least one value of $\boldsymbol{\theta} \in \mathcal{V}$ and where $\mathbf{MSE}(\mathbf{h}\{R(k)\}) = \mathbb{E}[\|\mathbf{h}\{R(k)\} - \boldsymbol{\theta}\|^2]$ with $\|\mathbf{y}\|^2 = \mathbf{y}^T\mathbf{y}$, $\mathbf{y} \in \mathbb{R}^{4q}$. There are many possible values of $\mathbf{A}$ and $\mathbf{b}$ that achieve (4) [21]. The strategy to be used here is the MiniMax approach developed by Y. C. Eldar [20]. Furthermore, a closed form expression for this estimator will be presented.

The main objective of this paper is to improve the estimation of the parameters of multiple damped complex exponentials through the use of the minimax affine estimator applied to the unbiased FML estimator.

In section 2 the Cramér-Rao Lower Bound (CRLB) for the parameters of a single damped complex exponential will be obtained and the general CRLB will be shown, in section 3 the Fast Maximum Likelihood Estimator is presented and in section 4 a closed-form expression for the MiniMax Affine Estimator is given. In section 5 a simulated example is used to show the better performance of the MiniMax affine estimator and finally, in section 6, some conclusions of this work are stated.

2 The Cramér-Rao Lower Bound

In this section, the Cramér-Rao Lower Bound (CRLB) for any unbiased estimator of the parameters of a single damped complex exponential, $\boldsymbol{\theta}_i \in \mathbb{R}^4$, $i = 1, \ldots, q$ is obtained. This will be used as a motivation for presenting the CRLB for unbiased estimators of all the parameters of multiple damped complex exponentials, $\boldsymbol{\theta} \in \mathbb{R}^{4q}$, as obtained in [22]. The inverse of the Fisher Information Matrix will be used as the covariance matrix of the FML estimator as it is considered efficient for moderately high values of SNR ([4], p. 2249).

For the unbiased FML estimator, $\mathbf{h}\{R(k)\}$, the CRLB is, by definition [23,24],

$$\mathbf{MSE}\left(\mathbf{h}\{R(k)\}\right) = \mathbf{Var}\left(\mathbf{h}\{R(k)\}\right) \geq \mathbf{tr}\left(\mathbf{J}^{-1}\right) \tag{5}$$

where $\mathbf{J}$ is the Fisher Information Matrix (FIM) [25],

$$\begin{aligned} \mathbf{J} &= [J_{ij}]_{i,j=1}^{4q} \\ J_{ij} &= \mathbb{E}\left[\frac{\partial}{\partial\theta_i}\left(\ln\left(p_{\{R(k)\}}(\{r(k)\};\boldsymbol{\theta})\right)\right)\frac{\partial}{\partial\theta_j}\left(\ln\left(p_{\{R(k)\}}(\{r(k)\};\boldsymbol{\theta})\right)\right)\right] \\ &= -\mathbb{E}\left[\frac{\partial^2}{\partial\theta_i\partial\theta_j}\left(\ln\left(p_{\{R(k)\}}(\{r(k)\};\boldsymbol{\theta})\right)\right)\right] \end{aligned} \tag{6}$$

where $\mathbf{J} = [J_{ij}]_{i,j=1}^{4q}$ is a matrix of size $4q \times 4q$ whose elements are J_{ij}, the likelihood function for the sample $\{R(k)\}_{k=0}^{n-1}$ is $p_{\{R(k)\}}(\{r(k)\};\boldsymbol{\theta})$, $\{r(k)\}_{k=0}^{n-1}$ is a realization of the sample and θ_i, $i = 1, \ldots, 4q$ are the elements of the vector $\boldsymbol{\theta}$.

In the case when a single complex damped sinusoid is considered, $q = 1$, and model (1) turns into

$$R(k) = x_0(k) + N(k) \quad ; \quad x_0(k) = c_0 \mathrm{e}^{j\,\beta_0}\,\mathrm{e}^{(\alpha_0 + j\,\omega_0)k} \tag{7}$$

with $\boldsymbol{\theta}_0 = \left[\alpha_0\ c_0\ \omega_0\ \beta_0\right]^T \in \mathbb{R}^4$ and where $N(k)$ is discrete-time, circularly symmetric AWGN, with $\mathbb{E}[N(k)N^*(k+m)] = \sigma_0^2\delta(m)$, with σ_0^2 known. The probability density function (pdf) for a sample of size n is given by [26]

$$p_{\{R(k)\}}(\{r(k)\};\boldsymbol{\theta}_0) = \frac{1}{(\pi\sigma_0^2)^n}\exp\left(-\frac{1}{\sigma_0^2}\sum_{k=0}^{n-1}\left|r(k) - c_0\mathrm{e}^{j\,\beta_0}\,\mathrm{e}^{(\alpha_0+j\,\omega_0)k}\right|^2\right) \tag{8}$$

Applying logarithm to (8), evaluating it in the sample $\{R(k)\}_{k=0}^{n-1}$ considering that $R(k) = R_{\mathfrak{Re}}(k) + j\,R_{\mathfrak{Im}}(k)$ and discarding the terms that do not depend on the paramater $\boldsymbol{\theta}_0$,

$$\begin{aligned} u(\{R(k)\},\boldsymbol{\theta}_0) = &-\frac{c_0^2}{\sigma_0^2}\sum_{k=0}^{n-1}\mathrm{e}^{2\alpha_0 k} \\ &+ \frac{2c_0}{\sigma_0^2}\sum_{k=0}^{n-1}\mathrm{e}^{\alpha_0 k}\left(R_{\mathfrak{Re}}(k)\cos(\omega_0 k + \beta_0) + R_{\mathfrak{Im}}(k)\sin(\omega_0 k + \beta_0)\right) \end{aligned} \tag{9}$$

is obtained. Then, differentiating twice with respect to each parameter and taking the opposite of the expected value, using the fact that $\mathbb{E}[R_{\mathfrak{Re}}(k)] = c_0 e^{\alpha_0 k} \cos(\omega_0 k + \beta_0)$ and $\mathbb{E}[R_{\mathfrak{Im}}(k)] = c_0 e^{\alpha_0 k} \sin(\omega_0 k + \beta_0)$, the elements of the Fisher Information Matrix (FIM) are

$$\begin{aligned}
-\mathbb{E}\left[\frac{\partial^2 u}{\partial \alpha_0^2}\right] &= 2\frac{c_0^2}{\sigma_0^2}\sum_{k=0}^{n-1} k^2 e^{2\alpha_0 k} \\
-\mathbb{E}\left[\frac{\partial^2 u}{\partial \alpha_0\, \partial c_0}\right] &= 2\frac{c_0}{\sigma_0^2}\sum_{k=0}^{n-1} k e^{2\alpha_0 k} \\
-\mathbb{E}\left[\frac{\partial^2 u}{\partial c_0^2}\right] &= \frac{2}{\sigma_0^2}\sum_{k=0}^{n-1} e^{2\alpha_0 k} \\
\mathbb{E}\left[\frac{\partial^2 u}{\partial \alpha_0\, \partial \omega_0}\right] &= \mathbb{E}\left[\frac{\partial^2 u}{\partial \alpha_0\, \partial \beta_0}\right] = \mathbb{E}\left[\frac{\partial^2 u}{\partial c_0\, \partial \omega_0}\right] = \mathbb{E}\left[\frac{\partial^2 u}{\partial c_0\, \partial \beta_0}\right] = 0 \\
-\mathbb{E}\left[\frac{\partial^2 u}{\partial \omega_0^2}\right] &= 2\frac{c_0^2}{\sigma_0^2}\sum_{k=0}^{n-1} k^2 e^{2\alpha_0 k} \\
-\mathbb{E}\left[\frac{\partial^2 u}{\partial \omega_0\, \partial \beta_0}\right] &= 2\frac{c_0^2}{\sigma_0^2}\sum_{k=0}^{n-1} k e^{2\alpha_0 k} \\
-\mathbb{E}\left[\frac{\partial^2 u}{\partial \beta_0^2}\right] &= 2\frac{c_0^2}{\sigma_0^2}\sum_{k=0}^{n-1} e^{2\alpha_0 k}
\end{aligned}$$

Therefore, the FIM is

$$\mathbf{J} = \begin{bmatrix}
\frac{2c_0^2}{\sigma_0^2}\sum\limits_{k=0}^{n-1} k^2 e^{2\alpha_0 k} & \frac{2c_0}{\sigma_0^2}\sum\limits_{k=0}^{n-1} k e^{2\alpha_0 k} & 0 & 0 \\
\frac{2c_0}{\sigma_0^2}\sum\limits_{k=0}^{n-1} k e^{2\alpha_0 k} & \frac{2}{\sigma_0^2}\sum\limits_{k=0}^{n-1} e^{2\alpha_0 k} & 0 & 0 \\
0 & 0 & \frac{2c_0^2}{\sigma_0^2}\sum\limits_{k=0}^{n-1} k^2 e^{2\alpha_0 k} & \frac{2c_0^2}{\sigma_0^2}\sum\limits_{k=0}^{n-1} k e^{2\alpha_0 k} \\
0 & 0 & \frac{2c_0^2}{\sigma_0^2}\sum\limits_{k=0}^{n-1} k e^{2\alpha_0 k} & \frac{2c_0^2}{\sigma_0^2}\sum\limits_{k=0}^{n-1} e^{2\alpha_0 k}
\end{bmatrix} \quad (10)$$

and the CRLB can be readily obtained as CRLB $= \mathbf{tr}(\mathbf{J}^{-1})$.

It is interesting to observe that the absolute parameters of $x_0(k)$ (i.e. the parameters that appear in the absolute value of $x(k)$, α_0 and c_0) are not related to the argument parameters in the CRLB, ω_0 and β_0. This feature will be further exploited in the simulations. Another feature of the CRLB $= \mathbf{tr}(\mathbf{J}^{-1})$ is that it is directly proportional to the noise variance, σ_0^2 and it decreseas as c_0 increases.

An important practical problem stems from (10). The FIM (and, consequently, the CRLB) depends on the damping factor α_0 and the amplitude of the signal c_0 which are unknown parameters. In order to overcome this difficulty, in practice, the estimated values of the parameters will be used instead.

To close this section, the inverse of the FIM for the general model (1) is presented. The proof of this result can be found in [22], p. 879.

$$\mathbf{J}^{-1} = \sigma^2 \mathbf{S}^{-1} \tilde{\mathbf{Q}} \mathbf{S}^{-1} \tag{11}$$

where

$$\tilde{\mathbf{Q}} = \left(2\mathfrak{Re}\{\mathbf{Z}\mathbf{Z}^H\}\right)^{-1} \in \mathbb{R}^{4q \times 4q}$$
$$\mathbf{S} = \mathrm{diag}\left[\boldsymbol{\Lambda}\ \mathbf{I}\ \boldsymbol{\Lambda}\ \boldsymbol{\Lambda}\right] \in \mathbb{R}^{4q \times 4q}$$

where the superscript H denotes complex conjugate, and where

$$\boldsymbol{\Lambda} = \mathrm{diag}\left[c_1 \ldots c_q\right] \in \mathbb{R}^{q \times q}$$
$$\mathbf{Z} = \begin{bmatrix} -\boldsymbol{\Theta}\,\mathbf{Z}'_N \\ \boldsymbol{\Theta}\,\mathbf{Z}_N \\ j\,\boldsymbol{\Theta}\,\mathbf{Z}'_N \\ j\,\boldsymbol{\Theta}\,\mathbf{Z}_N \end{bmatrix} \in \mathbb{C}^{4q \times n}$$

with

$$\boldsymbol{\Theta} = \mathrm{diag}\left[\mathrm{e}^{j\,\beta_1} \ldots \mathrm{e}^{j\,\beta_q}\right] \in \mathbb{C}^{q \times q}$$
$$\mathbf{Z}_N = \begin{bmatrix} 1\ \mathrm{e}^{(\alpha_1 + j\omega_1)} & \ldots & \mathrm{e}^{(\alpha_1 + j\omega_1)(n-1)} \\ \vdots\ \ \vdots & \ddots & \vdots \\ 1\ \mathrm{e}^{(\alpha_q + j\omega_q)} & \ldots & \mathrm{e}^{(\alpha_q + j\omega_q)(n-1)} \end{bmatrix} \in \mathbb{C}^{q \times n}$$
$$\mathbf{Z}'_N = \begin{bmatrix} 0\ \mathrm{e}^{(\alpha_1 + j\omega_1)} & \ldots & (n-1)\mathrm{e}^{(\alpha_1 + j\omega_1)(n-1)} \\ \vdots\ \ \vdots & \ddots & \vdots \\ 0\ \mathrm{e}^{(\alpha_q + j\omega_q)} & \ldots & (n-1)\mathrm{e}^{(\alpha_q + j\omega_q)(n-1)} \end{bmatrix} \in \mathbb{C}^{q \times n}$$

3 The Fast Maximum Likelihood Estimator

The Fast Maximum Likelihood (FML) estimator was developed by S. Umesh and D. W. Tufts and will be used as an unbiased, efficient estimator [4] of the parameters of model (1). The most relevant aspect of this work is that, by exploiting the particular *ridge* format of the compressed likelihood function (CLF), the search for the maximum value is reduced to $1-\mathrm{D}$ searches that are computationally more efficient than higher dimensional searches [4].

In order to obtain the algorithm that leads to the FML estimator $\mathbf{h}\{R(k)\}$, equation (1) has to be rewritten in matrix notation,

$$\mathbf{r} = \mathbf{X}(\boldsymbol{\alpha}, \boldsymbol{\omega})\mathbf{a} + \mathbf{n} \tag{12}$$

where $\mathbf{r} = \left[R(0) \ldots R(n-1)\right]^T$, $\boldsymbol{\alpha} = \left[\alpha_1 \ldots \alpha_q\right]^T$, $\boldsymbol{\omega} = \left[\omega_1 \ldots \omega_q\right]^T$,

$$\mathbf{X}(\boldsymbol{\alpha}, \boldsymbol{\omega}) = \begin{bmatrix} 1 & 1 & \ldots & 1 \\ \mathrm{e}^{\alpha_1 + j\,\omega_1} & \mathrm{e}^{\alpha_2 + j\,\omega_2} & \ldots & \mathrm{e}^{\alpha_q + j\,\omega_q} \\ \vdots & \vdots & \ddots & \vdots \\ \mathrm{e}^{(\alpha_1 + j\,\omega_1)(n-1)} & \mathrm{e}^{(\alpha_2 + j\,\omega_2)(n-1)} & \ldots & \mathrm{e}^{(\alpha_q + j\,\omega_q)(n-1)} \end{bmatrix}, \tag{13}$$

$\mathbf{a} = \left[c_1 \mathrm{e}^{j\,\beta_1} \ldots c_q \mathrm{e}^{j\,\beta_q}\right]^T$ and $\mathbf{n} = \left[N(0) \ldots N(n-1)\right]^T$.

It is observed that, while parameters $\boldsymbol{\alpha}$ and $\boldsymbol{\omega}$ appear as a nonlinear function in $\mathbf{X}$, parameters of complex amplitude $\mathbf{a}$ are linear, so that if the maximum likelihood estimators (MLE) of $\boldsymbol{\alpha}$ and $\boldsymbol{\omega}$ are available $(\hat{\boldsymbol{\alpha}}, \hat{\boldsymbol{\omega}})$, then

$$\hat{\mathbf{a}} = \left(\mathbf{X}^H(\hat{\boldsymbol{\alpha}}, \hat{\boldsymbol{\omega}})\mathbf{X}(\hat{\boldsymbol{\alpha}}, \hat{\boldsymbol{\omega}})\right)^{-1} \mathbf{X}^H(\hat{\boldsymbol{\alpha}}, \hat{\boldsymbol{\omega}})\,\mathbf{r} \tag{14}$$

because it is the solution of a least squares problem ([27], chap. 11, p. 483) and because of the invariance property of the MLE ([28], theo. 7.2, p. 176). This reduces the CLF ([29], p. 234) to

$$L(\boldsymbol{\alpha}, \boldsymbol{\omega}) = \mathbf{r}^H\,\mathbf{X}\left(\mathbf{X}^H\mathbf{X}\right)^{-1}\mathbf{X}^H\,\mathbf{r} \tag{15}$$

and therefore, the maximum likelihood estimates $(\hat{\boldsymbol{\alpha}}, \hat{\boldsymbol{\omega}})$ are obtained as ([4], p. 2246)

$$(\hat{\boldsymbol{\alpha}}, \hat{\boldsymbol{\omega}}) = \arg\max_{\boldsymbol{\alpha}, \boldsymbol{\omega}} L(\boldsymbol{\alpha}, \boldsymbol{\omega}) = \arg\max_{\boldsymbol{\alpha}, \boldsymbol{\omega}} \mathbf{r}^H\,\mathbf{X}\left(\mathbf{X}^H\mathbf{X}\right)^{-1}\mathbf{X}^H\,\mathbf{r} \tag{16}$$

By analizing the noiseless single damped complex exponential case, the authors realized that the CLF has a ridge structure that allows for two $1-\mathrm{D}$ searches, separately for α and for ω instead of carrying out a $2-\mathrm{D}$ search in the (α, ω) plane ([4], sec. II, p. 2246). The authors observed that the CLF for this particular case (7), has a ridge that runs through the true value of ω independently of the value of α. Therefore, for any fixed initial value of α,

$$(\hat{\alpha}, \hat{\omega}) = \arg\max_{\alpha, \omega} L_1(\alpha, \omega) = \arg\max_{\alpha, \omega} \mathbf{r}^H\mathbf{x}(\alpha, \omega)\left(\mathbf{x}^H(\alpha, \omega)\mathbf{x}(\alpha, \omega)\right)^{-1}\mathbf{x}^H(\alpha, \omega)\mathbf{r} \tag{17}$$

can be maximized with respect to ω and, after that, it can be maximized for α ([4], p. 2248), obtaining $(\hat{\alpha}, \hat{\omega})$. In (17), $\mathbf{x}(\alpha, \omega) = \left[1\ \mathrm{e}^{\alpha + j\,\omega} \ldots \mathrm{e}^{(\alpha + j\,\omega)(n-1)}\right]^T$.

Because this ridge structure of the CLF is useful only for the single damped complex exponential case, the authors proposed a modified Costas' residual signal analysis (RSA) [30] algorithm to separate the different damped complex exponentials and apply the corresponding single case search to each one of them.

Succinctly, the algorithm can be described as follows (for a mathematically detailed description, see [4], sec. III, p. 2249). First, for the initialization, the whole signal $\mathbf{r}$ (12) is considered as a single damped complex exponential so that the estimates $(\hat{\alpha}_1, \hat{\omega}_1)$ of the strongest signal are obtained using (17). With these estimates, an initial value of $\hat{a}_1$ is obtained by using (14). With all these four parameters, the strongest signal is reconstructed and then, substracted from the original signal $\mathbf{r}$. This allows for the second strongest component to reveal itself as the maximum. A new search is performed so that $(\hat{\alpha}_2, \hat{\omega}_2)$ is obtained and then, these estimates together with the previous ones $(\hat{\alpha}_1, \hat{\omega}_1)$ are used to obtain $\hat{a}_2$ and refine $\hat{a}_1$. Now, both the strongest components are reconstructed and substracted from $\mathbf{r}$. This continues until the parameters of the weakest signal $\boldsymbol{\theta}_q$ are estimated.

Once all the initial estimates of the parameters are obtained, the iterative improvement begins. All the previously estimated parameters are used to reconstruct all the components. These components are substracted from $\mathbf{r}$ with the exception of the first component. Then, a signal that is composed only of the first component is obtained, and the values of $(\hat{\alpha}_1, \hat{\omega}_1)$ are obtained through the use of (17). These two new estimates, together with all the other estimates are used in (14) to obtain a new estimate of $\hat{a}_1$. Then, all the components are reconstructed and substracted from $\mathbf{r}$ with the exception of the second component. The parameters (α, ω) of the second component are estimated and, after that, new estimates for $\mathbf{a}_i$ are obtained. Later, the reconstruction of all signals is carried out and substracted from $\mathbf{r}$ with the exception of the third component. This proceedure goes on for every component of the signal $\mathbf{r}$. All of this stage (the refinement of the q parameters $\boldsymbol{\theta}_i$, $i = 1, \ldots, q$) is repeated until a certain stop criteria is met (typically, the number of iterations).

The result of this algorithm yields the unbiased FML estimator $\mathbf{h}\{R(k)\}$.

4 The MiniMax Affine Estimator

In continuation to an earlier work [31], the affine estimation of parameters is expanded to the multidimensional case in order to apply it to the estimation of multiple damped complex exponentials parameters. In this section, the closed form expression of the MiniMax affine estimator is presented and some considerations about it are developed.

Let $\mathbf{h}\{R(k)\} \in \mathbb{R}^{4q}$ be an unbiased estimator of parameter $\boldsymbol{\theta} \in \mathbb{R}^{4q}$ based on sample $\{R(k)\}$. Let $\mathbb{E}[(\mathbf{h}\{R(k)\} - \boldsymbol{\theta})(\mathbf{h}\{R(k)\} - \boldsymbol{\theta})^T] = \boldsymbol{\Sigma}_{\mathbf{h}} \in \mathbb{R}^{4q \times 4q}$ be the constant symmetric positive-definite (s.p.d.) covariance matrix of the unbiased estimator and let the validation region $\mathcal{V}$ be an ellipsoid described by

$$\mathcal{V} = \left\{ \boldsymbol{\theta} \in \mathbb{R}^{4q} : (\boldsymbol{\theta} - \boldsymbol{\theta}_c)^T \, \mathbf{F} \, (\boldsymbol{\theta} - \boldsymbol{\theta}_c) \leq 1 \right\} \tag{18}$$

which is called the *validation-ellipsoid*. Matrix $\mathbf{F} \in \mathbb{R}^{4q \times 4q}$ is s.p.d. and describes the dimensions of the ellipsoid and $\boldsymbol{\theta}_c \in \mathbb{R}^{4q}$ is its center. Both quantities are known and summarize the previous information on the parameters.

The affine estimator $\mathbf{h}_B\{R(k)\}$ is given by equation (3)

$$\mathbf{h}_B\{R(k)\} = \mathbf{A}\, \mathbf{h}\{R(k)\} + \mathbf{b}$$

and, as stated in the introduction, the objective is to find the values of $\mathbf{A}$ and $\mathbf{b}$ such that $\mathbf{h}_B\{R(k)\}$ is dominant over $\mathbf{h}\{R(k)\}$ for all the values of $\boldsymbol{\theta}$ in the region $\mathcal{V}$ (4)

$$\mathbf{MSE}(\mathbf{h}_B\{R(k)\}) \leq \mathbf{MSE}(\mathbf{h}\{R(k)\}) \quad , \quad \forall \boldsymbol{\theta} \in \mathcal{V}$$

with strict inequality for at least one value of $\boldsymbol{\theta} \in \mathcal{V}$.

The MiniMax strategy for choosing the optimal values of $\mathbf{A}$ and $\mathbf{b}$ is [20]

$$\underset{\mathbf{A},\mathbf{b}}{\text{minimize}} \; \max_{\boldsymbol{\theta} \in \mathcal{V}} \{\mathbf{MSE}(\mathbf{h}_B\{R(k)\}) - \mathbf{MSE}(\mathbf{h}\{R(k)\})\} \tag{19}$$

The closed form expression of the optimal solution is

$$\begin{aligned} \mathbf{I} - \mathbf{A}^{\star} &= \frac{\mathbf{tr}\left((\boldsymbol{\Sigma}_{\mathbf{h}}\mathbf{F}\boldsymbol{\Sigma}_{\mathbf{h}})^{\frac{1}{2}}\right)}{1 + \mathbf{tr}\left(\boldsymbol{\Sigma}_{\mathbf{h}}\mathbf{F}\right)} \left(\boldsymbol{\Sigma}_{\mathbf{h}}\mathbf{F}\boldsymbol{\Sigma}_{\mathbf{h}}\right)^{\frac{1}{2}} \boldsymbol{\Sigma}_{\mathbf{h}}^{-1} \\ \mathbf{b}^{\star} &= \left(\mathbf{I} - \mathbf{A}^{\star}\right)\boldsymbol{\theta}_c \end{aligned} \tag{20}$$

provided that

$$\mathbf{F} \leq \frac{\mathbf{I}}{\mathbf{tr}\left(\boldsymbol{\Sigma}_{\mathbf{h}}\right)} \tag{21}$$

where the matrix inequality represents that $\frac{\mathbf{I}}{\mathbf{tr}(\boldsymbol{\Sigma}_{\mathbf{h}})} - \mathbf{F}$ is symmetric non-negative definite (s.n.n.d.).

From a practical standpoint, if condition (21) fails, then it would be the situation when the measurement errors (given by $\boldsymbol{\Sigma}_{\mathbf{h}}$) are greater than the accuracy of the previous information given by the validation ellipsoid, therefore, it would make no practical sense to take the measurements as there is no possible improvement to the previous information.

Solution (20) is proved optimal because it satisfies the KKT conditions in ([20], eq. 58, p. 3831) which are necessary and sufficient optimality conditions as the problem (19) is convex [32].

5 Examples

In this section, two simulations are carried out in order to show the performance of the MiniMax affine estimator (20) when compared to the FML estimator (section 3). Both of these simulations involve the exact same parameters as those that appear in [4] so as to achieve a better comparison.

A sample of size $n = 25$ of $q = 2$ damped complex exponentials is considered. The unknown parameters of each component are $\alpha_1 = -0.1$, $c_1 = 1$, $\omega_1 = 2\pi\, 0,52$, $\beta_1 = 0$ and $\alpha_2 = -0.2$, $c_2 = 1$, $\omega_2 = 2\pi\, 0,42$, $\beta_2 = 0$ ([4], p. 2253).

As observed in section 2, the CRLB for parameters β_i and ω_i, $i = 1, 2$ and the CRLB for the real parameters c_i and α_i, $i = 1, 2$ have no influence on each other. Also, because of their circular nature, both the frecuency and phase parameters are inherently restricted to the range $[-\pi, \pi)$ making them excellent choices for the application of the affine estimator. Therefore, the affine estimator in these examples will be applied to the frequency and phase unbiased estimators and the improvement on the mean squared error will be compared with respect to these four parameters $\omega_1, \beta_1, \omega_2, \beta_2$. In other words, the **MSE** is obtained as $\mathbf{MSE} = \mathbb{E}[\|\tilde{\boldsymbol{\theta}} - \tilde{\mathbf{h}}\{R(k)\}\|^2]$ where $\tilde{\boldsymbol{\theta}} = \left[\beta_1\ \omega_1\ \beta_2\ \omega_2\right]^T \in \mathbb{R}^4$ and $\tilde{\mathbf{h}}\{R(k)\}$ is the FML estimator for these four parameters. The **MSE** for the affine estimator is calculated analogously.

The validation-ellipsoid (18) for the parameter vector is given by

$$\mathbf{F} = \begin{bmatrix} 2.0507 & 1.9493 & 0 & 0 \\ 1.9493 & 2.0507 & 0 & 0 \\ 0 & 0 & 2.0507 & 1.9493 \\ 0 & 0 & 1.9493 & 2.0507 \end{bmatrix} \quad , \quad \boldsymbol{\theta}_c = \begin{bmatrix} 0 \\ 0.5252 \\ 0.0100 \\ 0.4158 \end{bmatrix}$$

The covariance matrix of the unbiased estimator is considered to be the inverse of the FIM for the phase and frequency, that is $\boldsymbol{\Sigma}_{\mathbf{h}} \approx \mathbf{J}^{-1}$. The values of α_1, α_2, c_1 and c_2 used in $\mathbf{J}^{-1}$ are the unbiased estimates obtained from the FML estimator. The iterative improvement of the FML algorithm consists of 15 iterations, the same as in [4].

For the simulations, the value of the SNR is varied from 0 dB to 30 dB with steps of 5 dB. For each value of SNR, 500 simulations are averaged in order to obtain estimates of the **MSE** as in [4].

The results are shown in figure 1. It is observed, from the simulations (fig. 1)

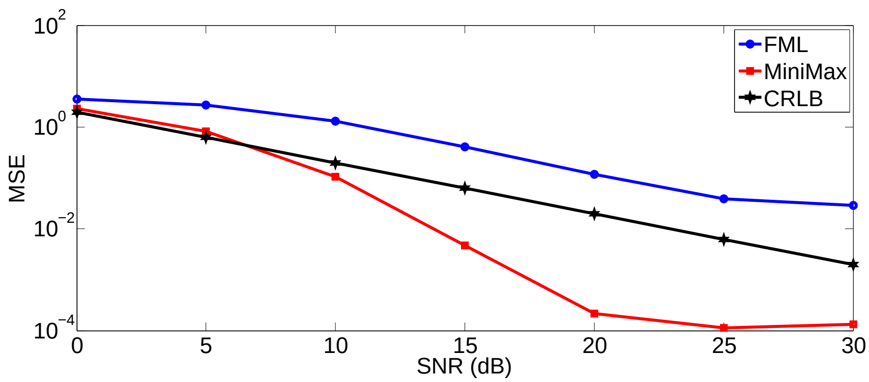

Fig. 1. MSE for frequency and phase estimates of both damped complex exponentials as a function of SNR for a sample of size $n = 25$.

that the MiniMax affine estimator results in a better performance when estimating frequency and phase. It has to be taken into consideration that for values of SNR ≤ 10 dB, the matrix $\mathbf{F}$ does not meet condition (21), so that the estimator is unreliable. Still, it performs better than the FML estimator.

For the second simulation, the same two damped complex exponentials are considered with the exception that for this case, the damping factors are quite different, making one exponential slowly decaying while the other decays faster. For this second simulation, $\alpha_1 = -0.07$ and $\alpha_2 = -0.4$ ([4], p. 2254). All the other values of the simulation remain the same. Results are shown in figure 2. It is observed that, for this case too, the MiniMax affine estimator performs

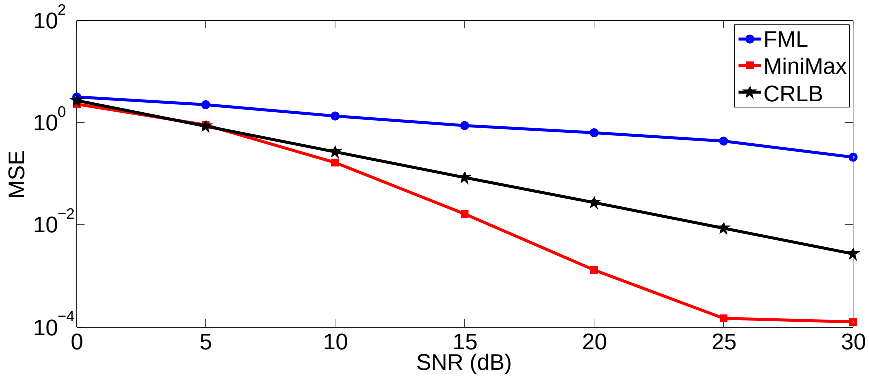

Fig. 2. MSE as a function of SNR for a sample of size $n = 25$ for two differently decaying exponentials.

better than the FML estimator. Also, taking into account that for SNR $\leq 10\,\mathrm{dB}$ condition (21) is not met, the affine estimator is better in all the range of values of SNR considered.

6 Conclusions

In this paper, the MiniMax affine estimator was used to improve the parameter estimation (frequency and phase) of multiple damped complex exponentials in AWGN. Also, a closed form expression for the MiniMax Affine Estimator was presented.

It was shown, through simulation, that the MiniMax affine estimator performs better than the FML, especially for low values of SNR, making it an excellent choice when the situation is adverse (in terms of SNR).

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