New facets of the 2-dominating set polytope of trees

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Abstract. Given a graph G and a nonnegative integer number k, a k-dominating set in G is a subset of vertices D such that every vertex in the graph is adjacent to at least k elements of D. The k-dominating set polytope is the convex hull of the incidence vectors of k-dominating sets in G. This is a natural generalization of the well-known dominating set polytope in graphs. In this work we study the 2-dominating set polytope of trees and we will provide new facet defining inequalities for it.

Keywords: 2-domination, facet, tree

1 Introduction and preliminaries results

It is known that domination and its variations arise in many applications, and that these problems are, in general, NP-hard problems [4]. However, domination in graphs has been shown to be polynomial time solvable in several graph classes such as (chordless) cycles [2], cactus graphs [5] and series-parallel graphs [6]. In this work we consider a generalization of this problem called *k-domination* in graphs, in particular we will focus in the 2-domination problem for trees.

If $G = (V, E)$ is a graph, we denote by $N_1(i)$ the *neighborhood* of i, i.e., the set of vertices of V adjacent to vertex i in G, $N_1[i] = N_1(i) \cup \{i\}$ stands for its *closed neighborhood* and $gr(i) = |N_1(i)|$. The minimum cardinality of all neighbourhoods of vertices of G is $\delta(G)$.

Given $G = (V, E)$ and an integer $k \leq \delta(G) + 1$, a *k-dominating set* in G is a set $S \subset V$ such that $|S \cap N_1[i]| \geq k$ for every vertex $i \in V$ (this definition was first introduced in [3] under the name of *k-tuple dominating set*). The minimum cardinality of a k-dominating set in G is denoted by $\gamma_k(G)$.

If $N_1[G]$ is the $\{0,1\}$-matrix whose rows are the incidence vectors of $N_1[i]$ for every $i \in V$ and $\mathbf{1}$ is the vector of all ones (of appropriate dimension), the *k-dominating set polytope* is defined as

$$Q_k^*(G) = \operatorname{conv}\left(\{x \in \{0,1\}^V : \ N_1[G]x \geq k\mathbf{1}\}\right).$$

Although in some particular cases all these inequalities also define facets, given an arbitrary graph G, it is not known te complete descripton of its k-dominating set polytope. In this work we will study the facets of the 2-dominating set polytope of a tree.

In order to do so, let us introduce the following concepts: let $M \in \{0,1\}^{m\times n}$ matrix having at least two 1's per row. If $ax \geq b$ is a valid inequality of $P(M) = \text{conv}\left(\{x \in \{0,1\}^n : \ Mx \geq \mathbf{c}\}\right)$, where $\mathbf{c} \in \mathbb{Z}_+^m$, we say that a pair of columns i, j is a-critical if there exist two roots X_1 and X_2 of $ax \geq b$ (i.e. integer points in $P(M)$ satisfying $ax = b$) such that

$$(X_1 - X_2)_k = \begin{cases} 0 & k \neq i,j \\ 1 & k = i \\ -1 & k = j \end{cases} \qquad (X_2 - X_1)_k = \begin{cases} 0 & k \neq i,j \\ -1 & k = i \\ 1 & k = j \end{cases}$$

The a- critical graph of M, denoted by $G^a(M)$ is the graph whose vertices are the columns of M and the edges are the a-critical pairs.

We will say that $U \subset \{1,...,n\}$ is a-connected if the subgraph of $G^a(M)$ induced by U is connected.

With these definitions, we can prove a result that generalizes a characterization of the facets of the set covering polyhedron defined by a matrix M, given in [1].

Theorem 1. *Let $M \in \{0,1\}^{m\times n}$ matrix having at least two 1's per row. Let $ax \geq b$ be a valid inequality of $P(M)$ con $b > 0$. Let $V_1, \cdots, V_p$ be a partition of $\{1,...,n\}$ such that V_i is a-connected for all $i \in \{1,...,p\}$. The inequality $ax \geq b$ is a facet of $P(M)$ if and only if there exist p roots $X_1, \cdots, X_p$ satisfying*

$$\begin{vmatrix} |X_1 \cap V_1| & \cdots & |X_1 \cap V_p| \\ \vdots & & \vdots \\ |X_p \cap V_1| & \cdots & |X_p \cap V_p| \end{vmatrix} \neq 0.$$

Applying the result above, in the following section we will give new facets of the 2-dominating set polytope of a tree.

2 The 2-dominating set polytope of trees

Let G be a tree. If L denotes the set of leaves of G and $N_1(L)$ the neighbors of the vertices in L. As the vertices in $i \in L \cup N_1(L)$ belong to every 2-dominating set of G, it is immediate that $dim(Q_2^*(G)) = |V - (L \cup N_1(L))|$.

The *trivial inequalities* $x_i \geq 0$ and $x_i \leq 1$, for all $i \in V$, the *neighborhood inequalities* $x(N_1[i]) \geq k$ for all $i \in V$are always valid for $Q_k^*(G)$. In the following results, we will characterize when they are also facet defining inequalities for the 2-dominating set polytope of a tree.

Firstly, as that each vertex in $L \cup N_1(L)$ belongs to every 2-dominating set of G, $x_i \leq 1$ is not a facet defining inequality of $Q_2^*(G)$. In fact, we can characterize the facets of the family $x_i \leq 1$ as follows.

Lemma 1. *Let G be a tree. A trivial inequality $x_i \leq 1$ is a facet defining inequality of $Q_2^*(G)$ if and only if $i \notin L \cup N_1(L)$.*

In addition, we have the following result.

Lemma 2. *Let G be a tree. A trivial inequality $x_i \geq 0$ is not a facet defining inequality of $Q_2^*(G)$.*

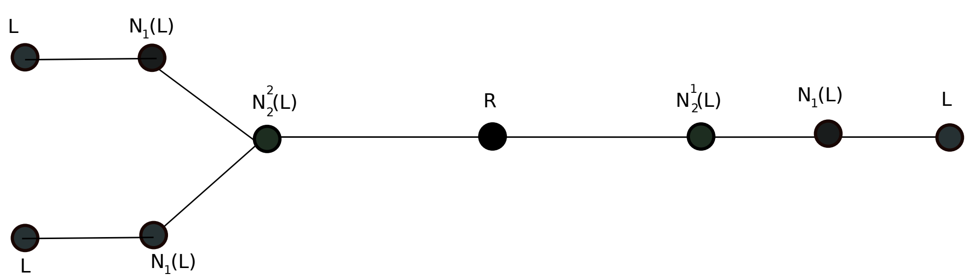
Now, as $x_i = 1$ holds if $i \in L \cup N_1(L)$, the system $N_1[G]x \geq 2$ can be rewritten as

$$\hat{N}[G]x \geq 2 \tag{1}$$

where $\hat{N}[G]$ is the row submatrix of $N_1[G]$ that contains the closed neighbourhoods of the vertices not in $L \cup N_1(L)$.

Moreover, if a vertex i has more than one neighbor in $N_1(L)$, the neighborhood inequality $x(N_1[i]) \geq 2$ is trivially satisfied and we denote by $N_2^2(L)$ the set of vertices in this class.

Let us denote by $N_2^1(L)$ the set of vertices not in $L \cup N_1(L)$ having one neighbor in $N_1(L)$, and let $R = V - (L \cup N_1(L) \cup N_2^1(L) \cup N_2^2(L))$ (see Figure).



Then, we can delete from the system $\hat{N}[G]x \geq 2$ the columns corresponding to the vertices in $L \cup N_1(L)$ and the rows corresponding to the closed neighborhoods of the vertices in $L \cup N_1(L) \cup N_2^2(L)$ and we obtain the system

$$\begin{cases} \hat{N}_1[G]x \geq 1 \\ \hat{N}_2[G]x \geq 2 \end{cases}$$

where $\hat{N}_1[G]$ ($\hat{N}_2[G]$) is the submatrix of $\hat{N}[G]$ that contains the closed neighborhoods of the vertices in $N_2^1(L)$ (resp., R), that is equivalent to (1).

Applying Theorem 1, we will study the neighborhood facets .

Case 1: $i \in N_2^1(L)$ In this case, the neighborhood facets can be characterized as follows:

Theorem 2. *Let G be a tree and $i \in N_2^1(L)$. The neighborhood inequality $x(N_1[i]) \geq 1$ is a facet defining inequality for $Q_2^*(G)$ if and only if*

1. *$gr(i) = 2$ and i has no neighbors in R having degree 2, or*
2. *$gr(i) \geq 3$ and i has no neighbors in $R \cup N_2^1(L)$ having degree 2.*

Case 2: $i \in R$ Let $i \in R$. From definition, $N[i] \subset R \cup N_2^1(L) \cup N_2^2(L)$.

The neighborhood inequality $x(N_1[i]) \geq 2$ is a valid inequality for $Q_2^*(G)$. Nevertheless, if $gr(i) \geq 3$ and $gr(j) = 2$ for all $j \in N(i)$, all the roots of $x(N_1[i]) \geq 2$ must contain vertex i, and then it is not a facet defining inequality for $Q_2^*(G)$.

Theorem 3. *Let $i \in R$. The neighborhood inequality $x(N_1[i]) \geq 2$ is a facet defining inequality for $Q_2^*(G)$ if and only if $gr(j) = 2$ for at most 2 vertices $j \in N(i)$.*

Now, if $i \in R$ and there are more than 3 vertices $j \in N(i)$ having degree 2, the neighborhood inequality $x(N_1[i]) \geq 2$ is not a facet defining inequality for $Q_2^*(G)$. In this case, we have the following result.

Theorem 4. *Let $i \in R$ such that $gr(j) \geq 3$ and $gr(j) = 2$ for all $j \in N(i)$. The*

$$(gr(i) - 1)x_i + \sum_{j \in N(i)} x_j \geq gr(i), \tag{2}$$

is a facet defining inequality for $Q_2^*(G)$.

3 Concluding remarks and a conjecture

A natural way of continuing our research is to fully characterize the 2-dominating set polytope of trees. Based on our computational experience, we claim that the 2-dominating set polytope of a tree G can be described using the following inequalities:

1. $x_i = 1$, if $i \in L \cup N_1(L)$,
2. $x_i \leq 1$, if $i \notin L \cup N_1(L)$,
3. the inequalities given in Theorem 2,
4. the inequalities given in Theorem 3, and
5. the inequalities given in Theorem 4.

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